K25P 2018

Reg. No. : ....

Name : ....

II Semester M.Sc. Degree (C.B.S.S.–Supplementary) Examination, April 2025 (2021 and 2022 Admissions) MATHEMATICS MAT 2C08 : Advanced Topology

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks. **(4×4=16)**

1. Prove that every totally bounded metric space is bounded.
2. Prove that every closed subset of a compact space is compact.
3. Prove that the Moor plane is not normal.
4. Let X = {1, 2, 3}. Find all topologies τ on X such that (X, τ) is regular.
5. Explain Hilbert cube.
6. Let (X, τ) be a topological space, let $x_0 \in$ X, and let $[\alpha] \in \pi_1$ (X, x_0). Then prove that there exists $[\overline{\alpha}] \in [\alpha] \in \pi_1$ (X, x_0) such that $[\alpha] \circ [\overline{\alpha}] = [\overline{\alpha}] \circ [\alpha] = [e]$.

PART – B

Answer **any four** questions from this part without omitting any Unit. **Each** question carries **16** marks. **(4×16=64)**

UNIT – I

7. Let (X, d) be a metric space. Then prove that the following statements are equivalent.
   a) (X, d) is compact.
   b) (X, d) is sequentially compact.
   c) (X, d) is countably compact.
   d) (X, d) has Bolzano-Weierstrass property.

P.T.O.

8. a) Prove that a topological space (X, τ) is compact if and only if every family of closed subsets of X with the finite intersection property has a nonempty intersection.

   b) Let (X, τ) be a topological space and let B be a basis for τ. Then prove that (X, τ) is compact if and only if every cover of X by members of B has a finite subcover.

   c) Let (X, τ) and (Y, U) be compact spaces. Then prove that $X \times Y$ is compact.

9. a) Prove that every closed subspace of locally compact Hausdorff space is locally compact.

   b) With suitable example, show that the continuous image of a locally compact space need not be locally compact.

   c) With detailed explanation, give an example of a topological space which has the Bolzano-Weierstrass property but it is not locally compact.

UNIT – II

10. a) Let (X, τ) be a topological space, let (Y, U) be a Hausdorff space, and let $f : X \to Y$ be continuous. Then prove that $\{(x_1, x_2) \in X \times X : f(x_1) = (x_2)\}$ is a closed subset of $X \times X$.

    b) For each i = 0, 1, 2 prove that the product of T_i- space is a T_i space.

11. a) Let $\{(X\alpha, \tau\alpha) : \alpha \in \Lambda\}$ be a family of topological spaces, and let $X = \prod_{\alpha \in \Lambda} X\alpha$. Then prove that (X, τ) is regular if and only if $(X\alpha, \tau\alpha)$ is regular for each $\alpha \in \Lambda$.

    b) Prove that a T_1 space (X, τ) is regular if and only if for each member p of X and each neighbourhood U of p, there is neighbourhood V of p such that $\overline{V} \subseteq U$.

    c) Prove that a T_1 space (X, τ) is regular if and only if for each $p \in X$ and each closed set C such that $p \notin C$, there exists an open sets U and V such that $C \subseteq U$, $p \in V$, and $\overline{U} \cap \overline{V} = \phi$.

12. a) Prove that every uncountable subset of a Lindelof space has a limit point.

    b) Prove that every second countable regular space is normal.

UNIT – III

13. State and prove Urysohn's Lemma.

14. Prove that a T_1 space (X, τ) is normal iff whenever A is closed subset of X and $f : A \to [-1, 1]$ is a continuous function, then there is a continuous function $F : X \to [-1, 1]$ such that $F|_A = f$.

15. a) Let (X, d) be a compact metric space, let (Y, U) be a Hausdorff space, and let $f : X \to Y$ be a continuous function that maps X onto Y. Then prove that (Y, U) is metrizable.

b) Let (X, τ) be a topological space, let $x_0 \in X$, and let $e : I \to X$ be the path defined by $e(x) = x_0$ for each $x \in I$. Then prove that $[\alpha] \circ [e] = [e] \circ [\alpha] = [\alpha]$ for each $[\alpha] \in \pi_1(X, x_0)$.